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Integral Form of the Radiative Transfer Equation Inside Refractive Cylindrical Media

Philippe Ben-Abdallah*
Université du Québec à Chicoutimi, Chicoutimi, Québec G7H 2B1, Canada
Sébastien Fumeron† and Vital Le Dez‡
Ecole Nationale Supérieure de Mécanique et d'Aérotechniques Chasseneuil du Poitou, 86960 Futuroscope, France
and
André Charette§
Université du Québec à Chicoutimi, Chicoutimi, Québec G7H 2B1, Canada

We consider the time-independent radiative transfer process inside cylindrical refractive semitransparent media with an internal space-dependent energy source, anisotropic scattering function, and specular reflecting boundary conditions. The integral form of the radiative transfer equation with any spatially continuous refractive index is worked out from its integro-differential form. To this end, the method of characteristics is used to solve the optical problem and, combined with a ray tracing technique, to integrate the radiative transfer equation along the curved light paths.

Nomenclature

A, a, B, b, C, c, D, d	=	partial curvilinear optical depth
$(e_r; e_\theta; e_z)$	=	cylindrical local base coordinates
$F(\tilde{\Omega}_I)$	=	external incident radiation, W-m^{-2}
g_α	=	radial function depending on the first invariant
H	=	optical Hamiltonian
h_α^β	=	radial function depending on both invariants α and β
I	=	intensity of radiation, W-m^{-2}
l	=	Lame coefficient ($l_1 = 1, l_2 = r, l_3 = 1$)
M_0	=	origin point along a trajectory
n	=	refractive index (real part)
$\mathcal{P}$	=	dissipated power, W-m^{-3}
p_j	=	partial derivative of the wave function with respect to cylindrical coordinates
$\tilde{p}_j$	=	modified derivative of the wave function with respect to cylindrical coordinates
Q	=	total angular source, W-m^{-3}
q_j	=	curvilinear coordinate ($q_1 = r, q_2 = \theta, q_3 = z$)
R	=	radius of the cylinder, m
r	=	radial distance to the origin, m
r_0	=	radius at the origin of the trajectory, m
r^*, r^{**}	=	root of h_α^β around r_0, m
S	=	spontaneous emission, W-m^{-3}
s	=	curvilinear abscissa, m
T	=	temperature at current point, K
T_R	=	temperature at the boundary, K
x_j	=	coordinates ($x_1 = x, x_2 = y, x_3 = z$)
α	=	first geodesic invariant
β	=	second geodesic invariant
γ	=	polar angle of the tangent vector
θ	=	azimuthal angle in the canonical base
κ	=	total lineic absorption field, m^{-1}
λ	=	wavelength, m
ρ	=	reflectivity of boundary surface
σ_s	=	inscattering probability density
$\tau_{s_A s_B}$	=	optical depth between two points A and B joined by a curved ray
φ	=	azimuthal angle of the tangent vector
ψ	=	wave function
Ω	=	unit vector tangent to the trajectory

Subscripts

I	=	at the impact point on the boundary
R	=	at the boundary
0	=	at the origin point
∞	=	external condition

Superscripts

$+$	=	upstream path
$-$	=	downstream path

Introduction

WHEN a wave spreads through a medium (a gas or dense media), it is well known that its associated electric field polarizes the molecules or the atoms and induces a radiative field, which interferes with the incident wave[1] with the result that it modifies its wave function. As this change is proportional to the depth of the medium traversed, it is equivalent to a change in its phase velocity. Also, generally speaking, within a medium the speed of a wave is a function of the position so that it streams in curved lines rather than straight lines. In thermal applications this phenomenon is generally neglected.

So far, only some two-dimensional radiative transfer problems with particular variation of the refractive index have been studied. These were mainly restricted to the study layered medium[2–4] and to some studies of radiative behavior of a slab with linear refractive index variation.[5,6] However, a general method applicable to the radiative transfer inside any refractive semitransparent media is not presently available. In this work we derive the integral forms of the radiative transfer equation for any cylindrical refractive semitransparent media. We consider (Fig. 1) an absorbing, emitting, and anisotropically scattering medium in radiative equilibrium with purely specular boundaries subjected to external incident radiation.

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*Researcher, Groupe de Recherche en Ingénierie des Procédés et Systèmes; currently at Centre National de la Recherche Scientifique.

†Ph.D. Student, Laboratoire d'Etudes Thermiques U.M.R. CNRS No. 6608.

‡Researcher, Laboratoire d'Etudes Thermiques U.M.R. CNRS No. 6608; currently at Centre National de la Recherche Scientifique.

§Professor, Groupe de Recherche en Ingéniere des Procédés et Systèmes.

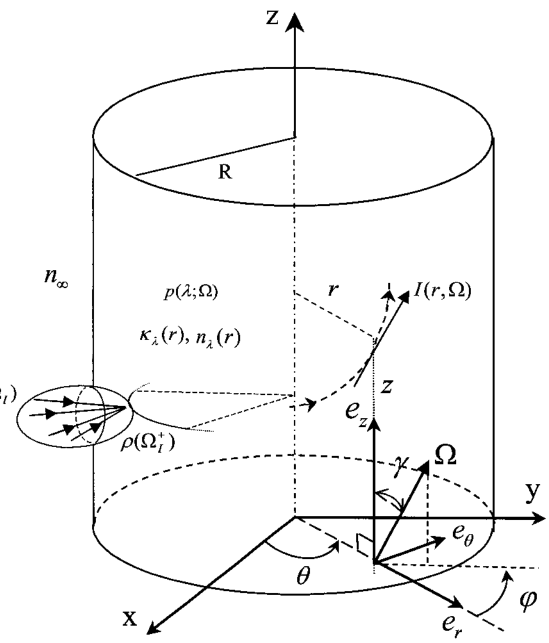

Fig. 1 Physical model and the geometrical meaning of the refractive angles (γ, φ).

The paper is organized as follows. In the first section we describe the basic equations that govern the optical behavior of any cylindrical refractive media with a spatially continuous variation of the refractive index. The method of characteristics is used to solve the optical problem, and both the equations of the light path and the corresponding metric are derived. Next, in order to integrate the radiative transfer equation inside a bounded medium, we use the notion of invariant geodesic to build the relations between the vector tangent to a light path at the boundary and its orientation at the point of origin. From these relations the specular and total internal reflection conditions are derived. Then the integral forms of the radiative transfer equations are established in the most general case, and numerical examples are presented to validate these theoretical results.

Basic Equations

Inside a refractive transparent medium[7] the ratio of the monochromatic intensity over the square of the refractive index $I(x, \Omega, \lambda)/n_\lambda^2(x)$ remains unchanged along any light path. Historically known as the Clausius principle, this result constitutes the basis of a phenomenological theory of the radiative transfer inside isotropic refractive semitransparent media. In fact, after introducing, at point x, emitting, absorbing, and scattering processes along a curved ray defined by its unit tangent Ω, the integro-differential form of the equation of transfer inside such a medium can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{I(x,\Omega,\lambda)}{n_\lambda^2(x)}\right] = Q(x,\Omega,\lambda) - \kappa_\lambda(x)\frac{I(x,\Omega,\lambda)}{n_\lambda^2(x)} \tag{1}$$

where

$$Q(x,\Omega,\lambda) = S_\lambda(x) + \int_0^{+\infty}\int_{4\pi}\sigma_s(\lambda'\to\lambda,\Omega'\to\Omega) \times \frac{I(x,\Omega',\lambda')}{n_{\lambda'}^2(x)}\,\mathrm{d}\Omega'\,\mathrm{d}\lambda' \tag{2}$$

Then, dropping the wavelength argument λ in order to simplify the notation and using

$$\exp\left\{\int_{s_0}^{s}\kappa[x(t)]\,\mathrm{d}t\right\}$$

as an integrating factor, we can solve Eq. (1) as

$$\frac{I(x,\Omega)}{n^2(x)} = \frac{I(x_0,\Omega_0)}{n^2(x_0)}\exp\left\{-\int_{s_0}^{s}\kappa[x(t)]\,\mathrm{d}t\right\} + \int_{s_0}^{s} Q[x(s'),\Omega(s')]\exp\left\{-\int_{s'}^{s}\kappa[x(t)]\,\mathrm{d}t\right\}\mathrm{d}s' \tag{3}$$

where s_0 is an arbitrary point along the path. It should be emphasized that this equality corresponds to the exact solution of the radiative transfer equation [Eq. (1)] in the special case of no scattering. On the other hand, in the general case, Eq. (3) is an integral form, which might be easily solved using, for instance, the Galerkin method,[8] the variational method,[9] or the collocation method.[10] In this work we will express Eq. (3) for any cylindrical refractive semitransparent media.

Because the ray trajectories (the geometrical optics approximation is assumed valid everywhere in the medium) coincide with the characteristic functions of the eikonal equation, they can be calculated from the latter. To this end, it is convenient[11] in the cylindrical case to integrate the eikonal equation in the following curvilinear form:

$$H(p_i, q_i) = \frac{1}{2}\left[\sum_{i=1}^{3}\frac{p_j^2}{l_j^2(q_i)} - n^2(q_i)\right] = 0 \tag{4}$$

where p_j denotes the partial derivative of the wave function ψ with respect to curvilinear coordinates q_j ($q_1 = r$, $q_2 = \theta$, $q_3 = z$), and l_j is the corresponding Lame coefficient ($l_1 = 1$, $l_2 = r$, $l_3 = 1$). Substitution of Eq. (4) into its following equations of characteristics[11]

$$\frac{\mathrm{d}q_j}{\mathrm{d}s} = \frac{\partial H}{\partial p_j}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}s} = -\frac{\partial H}{\partial q_j}, \qquad \frac{\mathrm{d}\psi}{\mathrm{d}s} = \sum_{j=1}^{3} p_j\frac{\partial H}{\partial p_j} \tag{5}$$

where $\mathrm{d}s$ denotes the differential element of a parameter (for instance, the curvilinear abscissa) along these characteristics, yields the ray equations

$$\frac{\mathrm{d}q_j}{\mathrm{d}s} = l_j^{-2}p_j, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}s} = n\frac{\partial n}{\partial q_j} + \sum_{i=1}^{3} l_i^{-3}\frac{\partial l_i}{\partial q_j}p_i^2 \tag{6}$$

Using in place of p_j the components in curvilinear coordinates

$$\tilde{p}_j = (1/l_j)p_j \tag{7}$$

the ray equation can be cast in an alternative form:

$$\frac{\mathrm{d}q_j}{\mathrm{d}s} = \frac{1}{l_j}\tilde{p}_j, \qquad \frac{\mathrm{d}\tilde{p}_j}{\mathrm{d}s} = \frac{1}{l_j}n\frac{\partial n}{\partial q_j} + \frac{1}{l_j}\sum_{i=1}^{3}\frac{\tilde{p}_i}{l_i}\left(\tilde{p}_i\frac{\partial l_i}{\partial q_j} - \tilde{p}_j\frac{\partial l_j}{\partial q_i}\right) \tag{8}$$

This leads to the following equalities in cylindrical coordinates:

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \tilde{p}_r, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{r}\tilde{p}_\theta, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \tilde{p}_z, \qquad \frac{\mathrm{d}\tilde{p}_r}{\mathrm{d}s} = n\frac{\mathrm{d}n}{\mathrm{d}r} + \frac{\tilde{p}_\theta^2}{r}$$

$$\frac{\mathrm{d}\tilde{p}_\theta}{\mathrm{d}s} = -\frac{1}{r}\tilde{p}_\theta\tilde{p}_r, \qquad \frac{\mathrm{d}\tilde{p}_z}{\mathrm{d}s} = 0 \tag{9}$$

Thus, from the last equality,

$$\tilde{p}_z = \tilde{p}_z^0 = \alpha \tag{10}$$

is constant along a ray. Moreover, as $\tilde{p}_r$, $\tilde{p}_\theta$, $\tilde{p}_z$ are related by the eikonal equation

$$\tilde{p}_r^2 + \tilde{p}_\theta^2 + \tilde{p}_z^2 = n^2(r) \tag{11}$$

we have the relation

$$r^2\tilde{p}_\theta^2 = r_0^2\tilde{p}_{\theta_0}^2 = \beta^2 \tag{12}$$

Then from Eqs. (10–12)

$$\tilde{p}_r = \pm\sqrt{n^2 - (\beta/r)^2 - \alpha^2} \tag{13}$$

Then, Eqs. (9), (10), and (13) lead to the following parametric ray equations:

$$\frac{d\theta}{dr} = \frac{\tilde{p}_\theta}{r\tilde{p}_r} = \pm\frac{\beta}{r\sqrt{n^2r^2 - \beta^2 - \alpha^2r^2}}$$

$$\frac{dz}{dr} = \frac{\tilde{p}_z}{\tilde{p}_r} = \pm\frac{\alpha r}{\sqrt{n^2r^2 - \beta^2 - \alpha^2r^2}} \tag{14}$$

The metric, which is necessary for integrating the radiative transfer equation (RTE) along any curvilinear path, is given by

$$ds = \left\|\frac{dx}{dr}\right\| dr = \left[1 + \left(\frac{dz}{dr}\right)^2 + r^2\left(\frac{d\theta}{dr}\right)^2\right]^{\frac{1}{2}} dr \tag{15}$$

because in cylindrical coordinates

$$dx = dr e_r + r\, d\theta e_\theta + dz\, e_z \tag{16}$$

Finally from Eqs. (14) and (15) we have

$$ds = \frac{n(r)r}{\sqrt{n^2(r)r^2 - \beta^2 - \alpha^2r^2}} dr \tag{17}$$

Invariant Geodesic and Reflecting Conditions

To deal with the boundary conditions for a light trajectory having variable local curvature and torsion inside a bounded medium, we have to relate the conditions encountered at the boundary to the ones already known. To this end, let us consider one point on a curved path light and denote Ω the unit tangent vector at this point of the curve. From the eikonal equation (11), it follows that

$$\Omega = \frac{1}{n}\begin{pmatrix}\tilde{p}_r\\ \tilde{p}_\theta\\ \tilde{p}_z\end{pmatrix} = \begin{pmatrix}\sin\gamma\cos\varphi\\ \sin\gamma\sin\varphi\\ \cos\gamma\end{pmatrix} \tag{18}$$

where γ and φ denote the refractive angle (Fig. 1), that is to say the polar angle and azimuthal angles, respectively, of the tangent vector in the local frame. Hence, from Eqs. (10), (12), and (18) we see that the couple (α, β), which defines the direction of the tangent at the origin of any path, is directly related to its direction at the current point by the two following relations:

$$\alpha = \tilde{p}_z = n(r)\cos\gamma = \tilde{p}_z^0 = n(r_0)\cos\gamma_0$$

$$\beta = r\tilde{p}_\theta = n(r)r\sin\gamma\sin\varphi = r_0\tilde{p}_\theta^0 = n(r_0)r_0\sin\gamma_0\sin\varphi_0 \tag{19}$$

The two parameters α and β are both invariant along the trajectory, which is fully defined by the initial location $(r_0;\ \gamma_0;\ \varphi_0)$. This concept of invariant geodesic is used to establish the reflection conditions on the boundary surface. It extends Bouguer's law[12] to the third dimension.

Now, let us consider, without restricting the generality of the reasoning, an arc of light that is spiralling up within the medium. When the trajectory of the light ray strikes the surface at the impact point I, the vector Ω^- tangent to the light path (Fig. 2) is specified in the local base by its polar angle γ^- and its azimuthal angle φ^-. From the invariant relations of Eq. (19), it follows that

$$n(r_0)\cos\gamma_0 = n(R)\cos\gamma^-$$

$$n(r_0)r_0\sin\gamma_0\sin\varphi_0 = n(R)R\sin\gamma^-\sin\varphi^- \tag{20}$$

The tangent vector Ω^+ of the reflected ray is determined by its polar angle γ^+ and its azimuthal angle φ^+. If the reflection is purely specular on the boundary (Fig. 2), the following relations between the incident and the reflected rays hold:

$$\gamma^+ = \gamma^-, \qquad \varphi^+ = \pi - \varphi^- \tag{21}$$

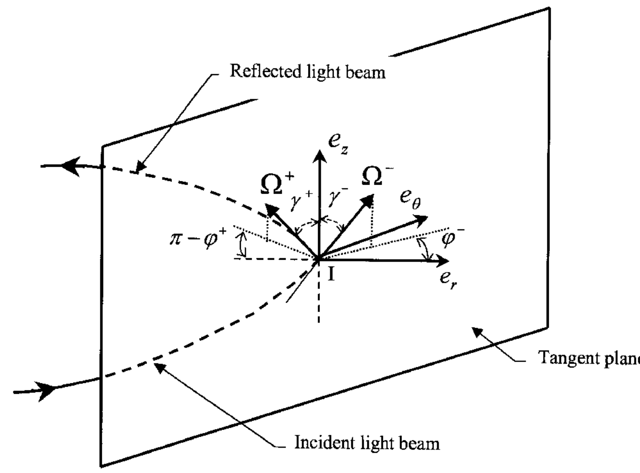

Fig. 2 Specular reflection on the boundary.

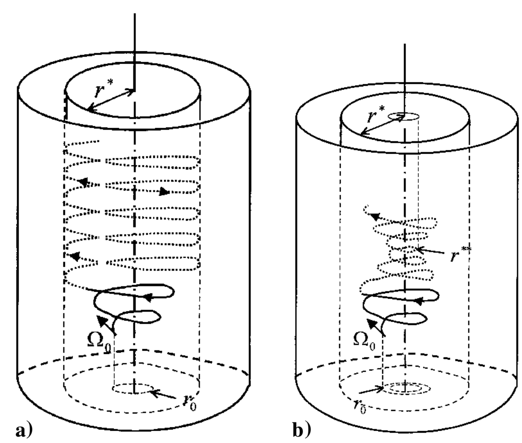

Fig. 3 Illustration of the path that the light takes from a point of total reflection. Both cases satisfy the ray equation but only b) satisfies the reciprocity theorem.

Hence, the reflection conditions can be written as

$$\alpha^+ = n(R)\cos\gamma^- = \alpha^- = \alpha$$

$$\beta^+ = n(R)R\sin\gamma^+\sin\varphi^+ = n(R)R\sin\gamma^-\sin\varphi^- = \beta^- = \beta \tag{22}$$

Thus the invariants are conserved after reflection at the boundary.

Now we will focus our attention on the internal reflection phenomenon, which might occur when the refractive index is spatially variable. To simplify the notation, it is convenient to introduce on the interval $[0;\ R]$ the following functions, which are dependent on parameters α and β:

$$g_\alpha(r) = n^2(r)r^2 - \alpha^2r^2, \qquad h_\alpha^\beta(r) = g_\alpha(r) - \beta^2 \tag{23}$$

When a ray reaches a point whose radius r^* is a root of h_α^β, then it passes through a total reflection point. In fact, from Eq. (14) we see that both $d\theta/dr$ and dz/dr are infinite in this case. Hence, the trajectory is locally tangent to a cylinder of radius r^*. But what happens from this point? Does the ray continue its path by spiralling up along this cylinder to follow an helical trajectory (Fig. 3a) or does it follow a spiral with a radius that changes with respect to z (Fig. 3b)? Mathematically both cases are acceptable because they both satisfy Eqs. (14). However only the case of Fig. 3b is physically admissible because it is the only one that includes light path reversibility in accordance with the reciprocity theorem (i.e., the path of the light is independent of sense propagation). Moreover because the ray must be tangent to a cylinder of radius r^*, at the total reflection point, and also conserve the geodesic invariant across this point, the following supplementary relations are valid:

$$\varphi^{*-} = \varphi^{*+} = \varphi^* = \pi/2, \qquad \gamma^{*-} = \gamma^{*+} = \gamma^* \tag{24}$$

$$\alpha^* = n(r^*)\cos\gamma^* = \alpha, \qquad \beta^* = n(r^*)r^*\sin\gamma^* = \beta \tag{25}$$

Integral Equations

Hereafter we will focus our attention on an origin point M_0 of coordinates $x_0 \equiv (r_0, \theta_0, z_0)$ strictly located inside this cylinder. We can distinguish four integral forms for the radiative transfer equation with respect to the initial conditions and the local behavior of h_α^β.

Case 1: $\gamma_0 \in]0; \pi/2[$(i.e., $\alpha > 0$) and $\varphi_0 \in]0; \pi/2[$(i.e., $\beta > 0$) (The reversed bracket means that the interval excludes the end points.)

Case 1a: $h_\alpha^\beta(r) = 0$ has no root on $]r_0; R]$

Because of the initial conditions, it is clear from M_0 that the light path spirals up with an increasing radius. Therefore, from Eq. (14) the trajectory is given by

$$\theta(r) = \theta_0 + \beta \int_{r_0}^{r} \frac{\mathrm{d}t}{t\sqrt{h_\alpha^\beta(t)}} \quad \text{on the interval } [r_0; R]$$

$$z(r) = z_0 + \alpha \int_{r_0}^{r} \frac{\mathrm{d}t}{\sqrt{h_\alpha^\beta(t)}} \tag{26}$$

When the radius is equal to R, the ray reaches the boundary of the cylinder and is specularly reflected. From impact point I the radius must decrease, whereas from Eq. (21) we know that z continues to increase. Therefore the trajectory is governed by

$$\theta(r) = \theta_I - \beta \int_{R}^{r} \frac{\mathrm{d}t}{t\sqrt{h_\alpha^\beta(t)}} \quad \text{on the interval } [R; r^*]$$

$$z(r) = z_I - \alpha \int_{R}^{r} \frac{\mathrm{d}t}{\sqrt{h_\alpha^\beta(t)}} \tag{27}$$

where r^* denotes the largest root of h_α^β on the interval $]0; r_0]$ while $(\theta_I; z_I)$ represents the polar coordinates of I, which are calculated from Eq. (26) for radius $r = R$. In accordance with the results derived in the preceding section concerning the total reflection conditions, an asymptotic case of a form similar to the one encountered in Fig. 3a must be discarded. Thus, integration of the RTE from Eq. (3) along the trajectory between the origin point and the impact point gives

$$\frac{I(x_0, \Omega_0)}{n_0^2} = \int_{s_0}^{s_I} Q(x, \Omega)\exp\left(-\tau_{s_0 s}\right)\mathrm{d}s + \rho\left(\Omega_I^+\right)\frac{I_1^-\left(R, \Omega_I^+\right)}{n_R^2} \times \exp\left(-\tau_{s_0 s_I}\right) + [1 - \rho(\tilde{\Omega}_I)]\frac{F(\tilde{\Omega}_I)}{n_\infty^2}\exp\left(-\tau_{s_0 s_I}\right) \tag{28}$$

where

$$\tau_{s_1 s_2} = \int_{s_1}^{s_2} \kappa[r(s)]\,\mathrm{d}s$$

and Ω_I^+ denote the curvilinear optical depth and the direction of the tangent vector of the reflected ray (Fig. 2) with respect to the internal normal at the impact point, respectively. Ω_I^+ is obviously related to the tangent vector of the ray at the boundary and can be calculated from both equalities in Eq. (20). $\tilde{\Omega}_I$ represents the angle of the incoming external flux with respect to the external normal, which contributes to the energy balance in the direction Ω_0. $\tilde{\Omega}_I$ is directly related to Ω_I^+ by the Fresnel relations.[12]

The physical meaning of Eq. (28) is clear. It represents the energy that comes in at the origin in the direction Ω_0, which may be decomposed into three parts. The first part, described by the first term of the right-hand side, concerns the contribution of the medium along the arc joining the origin to the boundary. The second part [second term of Eq. (28)] is related to the multireflection with ρ denoting the directional reflectivity on the boundary; the last part is caused by externally incident radiation. Using the relation in Eq. (17) to express the metric with respect to the radius and decomposing the last term of Eq. (28) by introducing the total reflection point, this equation can be cast as follows:

$$\begin{aligned}
\frac{I(x_0, \Omega_0)}{n_0^2} &= \int_{r_0}^{R} Q(x, \Omega)\exp\left(-\tau_{s_0 s}\right)\mathrm{d}s \\
&\quad + \rho\left(\Omega_I^+\right)\frac{I_1^-\left(R, \Omega_I^+\right)}{n_R^2}\exp(-a) + [1 - \rho(\tilde{\Omega}_I)]\frac{F(\tilde{\Omega}_I)}{n_\infty^2}\exp(-a) \\
&= \int_{r_0}^{R} Q(x, \Omega)\exp\left(-\tau_{s_0 s}\right)\mathrm{d}s + \rho\left(\Omega_I^+\right)\exp(-a) \\
&\quad \times \left\{\frac{I_1^+\left(R, \Omega_I^+\right)}{n_R^2}\exp(-2A) + \int_{r^*}^{R} Q(x, \Omega)\exp\left(-\tau_{s s_I}\right)\mathrm{d}s \right. \\
&\quad \left. + \int_{r^*}^{R} Q(x, \Omega)\exp(-\tau_{s^* s} - A)\,\mathrm{d}s(r)\right\} \\
&\quad + [1 - \rho(\tilde{\Omega}_I)]\frac{F(\tilde{\Omega}_I)}{n_\infty^2}\exp(-a)[\rho(\tilde{\Omega}_I)\exp(-2A) + 1]
\end{aligned} \tag{29}$$

where

$$A = \int_{r^*}^{R} \frac{\kappa(r)n(r)r\,\mathrm{d}r}{\sqrt{h_\alpha^\beta(r)}}, \qquad a = \int_{r_0}^{R} \frac{\kappa(r)n(r)r\,\mathrm{d}r}{\sqrt{h_\alpha^\beta(r)}} \tag{30}$$

represent the optical depth along the spiral arc, which relates the boundary to the total reflection point, and along the principal arc, which relates the origin to the boundary, respectively. I_i^- and I_i^+ represent the "downstream" and the "upstream" intensity along the arc, which will be subject to i successive reflections, whereas the last term represents (Fig. 1) the contribution of the externally incident radiation F. After some algebraic manipulation Eq. (29) becomes, for the mth order,

$$\begin{aligned}
\frac{I(x_0, \Omega_0)}{n_0^2} &= \int_{r_0}^{R} Q(x, \Omega)\exp\left(-\tau_{s_0 s}\right)\mathrm{d}s + \exp(-a) \\
&\quad \times \sum_{i=1}^{m} \rho^i\left(\Omega_R^+\right)\exp\{-[a + (2i-2)A]\} \\
&\quad \times \left[\int_{r^*}^{R} Q(x, \Omega)\exp\left(-\tau_{s s_I}\right)\mathrm{d}s \right. \\
&\quad \left. + \int_{r^*}^{R} Q(x, \Omega)\exp(-\tau_{s^* s} - A)\,\mathrm{d}s\right] + \rho^m\left(\Omega_R^+\right)\frac{I_m^+\left(R, \Omega_R^+\right)}{n_R^2} \\
&\quad \times \exp[-(a + 2mA)] + [1 - \rho(\tilde{\Omega}_I)]\frac{F(\tilde{\Omega}_I)}{n_\infty^2}\exp(-a) \\
&\quad \times \sum_{i=0}^{m} \rho^i(\tilde{\Omega}_I)\exp(-2iA)
\end{aligned} \tag{31}$$

Because $\rho < 1$ for any direction, the penultimate term of Eq. (31) vanishes when m (i.e., the number of reflections at the boundary) tends to infinity, and after evaluating the geometrical series one obtains

$$\frac{I(x_0,\Omega_0)}{n_0^2}=\int_{r_0}^{R}Q(x,\Omega)\exp\left(-\tau_{s_0s}\right)\mathrm{d}s+\frac{\rho\left(\Omega_R^+\right)\exp(-a)}{1-\rho\left(\Omega_R^+\right)\exp(-2A)}\times\left[\int_{r^*}^{R}Q(x,\Omega)\exp\left(-\tau_{ss_I}\right)\mathrm{d}s+\exp(-A)\int_{r^*}^{R}Q(x,\Omega)\exp(-\tau_{s^*s})\,\mathrm{d}s\right]+[1-\rho(\tilde{\Omega}_I)]\frac{F(\tilde{\Omega}_I)}{n_\infty^2}\frac{\exp(-a)}{1-\rho(\tilde{\Omega}_I)\exp(-2A)} \tag{32}$$

Case 1b: $h_\alpha^\beta(r)=0$ has at least one root on $]r_0;R]$.

Let r^* and r^{**} denote the smallest root of equation $h_\alpha^\beta=0$ on the interval $]r_0;R]$ and its largest root on $]0;r_0]$, respectively. It is clear that this type of trajectory never reaches the boundary of the cylinder so that the corresponding flux line is isolated from the external medium. However, the origin itself is not isolated from the external medium, because for other initial conditions (other values of β) the light path is connected to the boundary and the origin exchanges energy directly with the environment. The integration of the RTE in this case leads to

$$\frac{I(x_0,\Omega_0)}{n_0^2}=\int_{r_0}^{r^*}Q(x,\Omega)\exp\left(-\tau_{s_0s}\right)\mathrm{d}s+\exp(-b)\times\left\{\sum_{i=0}^{+\infty}\int_{r^{**}}^{r^*}Q(x,\Omega)\exp(-\tau_{ss^*}-2iB)\,\mathrm{d}s+\sum_{i=0}^{+\infty}\int_{r^{**}}^{r^*}Q(x,\Omega)\exp[-\tau_{s^{**}s}-(2i+1)B]\,\mathrm{d}s\right\} \tag{33}$$

where

$$B=\int_{r^{**}}^{r^*}\frac{\kappa(r)n(r)r\,\mathrm{d}r}{\sqrt{h_\alpha^\beta(r)}},\qquad b=\int_{r_0}^{r^*}\frac{\kappa(r)n(r)r\,\mathrm{d}r}{\sqrt{h_\alpha^\beta(r)}} \tag{34}$$

Thus

$$\frac{I(x_0,\Omega_0)}{n_0^2}=\int_{r_0}^{r^*}Q(x,\Omega)\exp\left(-\tau_{s_0s}\right)\mathrm{d}s+\frac{\exp(-b)}{1-\exp(-2B)}\times\left[\int_{r^{**}}^{r^*}Q(x,\Omega)\exp(-\tau_{ss^*})\,\mathrm{d}s+\exp(-B)\int_{r^{**}}^{r^*}Q(x,\Omega)\exp(-\tau_{s^{**}s})\,\mathrm{d}s\right] \tag{35}$$

Case 2: $\gamma_0\in]0;\pi/2[$ (i.e., $\alpha>0$) and $\varphi_0\in]\pi/2;\pi[$ (i.e., $\beta>0$).

Case 2a: $h_\alpha^\beta(r)=0$ has no root on $]r_0;R]$.

Let r^* denote the higher root of h_α^β on the interval $]0;r_0[$. Then, in a similar way to that followed in case 1a, by introducing

$$C=\int_{r^*}^{R}\frac{\kappa(r)n(r)r\,\mathrm{d}r}{h_\alpha^\beta(r)}=A,\qquad c=\int_{r^*}^{r_0}\frac{\kappa(r)n(r)r\,\mathrm{d}r}{h_\alpha^\beta(r)}=A-a \tag{36}$$

one obtains after a straightforward calculation

$$\frac{I(r_0,\Omega_0)}{n_0^2}=\int_{r^*}^{r_0}Q(x,\Omega)\exp\left(-\tau_{ss_0}\right)\mathrm{d}s+\int_{r^*}^{R}Q(x,\Omega)\exp(-\tau_{s^*s}-c)\,\mathrm{d}s+\frac{\rho\left(\Omega_R^+\right)\exp[-(c+C)]}{1-\rho\left(\Omega_R^+\right)\exp(-2C)}\left[\int_{r^*}^{R}Q(x,\Omega)\exp(-\tau_{s^*s}-C)\,\mathrm{d}s+\int_{r^*}^{R}Q(x,\Omega)\exp\left(-\tau_{ss_I}\right)\mathrm{d}s\right]+[1-\rho(\tilde{\Omega}_I)]\frac{F(\tilde{\Omega}_I)}{n_\infty^2}\frac{\exp[-(c+C)]}{1-\rho(\tilde{\Omega}_I)\exp(-2C)} \tag{37}$$

Case 2b: $h_\alpha^\beta(r)=0$ has at least one root on $]r_0;R]$.

By decomposing the intensity of the energy incoming at point r_0 from the elementary arcs (between the two total reflection points) in the same way as for case 1b, one obtains:

$$\frac{I(x_0,\Omega_0)}{n_0^2}=\int_{r^{**}}^{r_0}Q(x,\Omega)\exp\left(-\tau_{ss_0}\right)\mathrm{d}s+\frac{\exp(-d)}{1-\exp(-2D)}\times\left[\int_{r^{**}}^{r^*}Q(x,\Omega)\exp(-\tau_{s^{**}s})\,\mathrm{d}s+\exp(-D)\int_{r^{**}}^{r^*}Q(x,\Omega)\exp(-\tau_{ss^*})\,\mathrm{d}s\right] \tag{38}$$

where

$$D=\int_{r^{**}}^{r^*}\frac{\kappa(r)n(r)r\,\mathrm{d}r}{\sqrt{h_\alpha^\beta(r)}}=B$$

$$d=\int_{r^{**}}^{r_0}\frac{\kappa(r)n(r)r\,\mathrm{d}r}{\sqrt{h_\alpha^\beta(r)}}=B-b \tag{39}$$

The integral form, represented by Eqs. (32), (35), (37), and (38), of the radiative transfer equation (1) can be solved using one of the approximate methods[8–10] described in the Introduction. Although algebraically these relations appear very complex, their physical interpretation is nearly as simple as for the equations to nonrefractive semitransparent media. The main differences are caused by the decomposition of the light path in elementary arcs relating both reflection points on the boundary to inner reflection points and inner reflection points among themselves. In this latter case the rays never reach the boundary of the medium. Such a case never occurs inside typical semitransparent media.

Numerical Results

To examine the inherent effects linked to the nonlinear optical behavior of a refractive medium, we consider the following cases.

Case A: Curvature Effects

One considers a cylindrical fiber whose refractive index is a linear increasing function ($n=2+r/R$). In this case it is easy to show that the trajectories always are emerging because h_α^β has only one root. In comparison with the homogeneous case ($n=2.5$), the temperature field is greater at the center of the cylinder (Fig. 4). This phenomenon can easily be interpreted by analyzing the curvature of light paths within such a medium. In fact, because the refractive index is an increasing function, the paths are convex functions with respect to the radius (i.e., the rays bend toward the increasing refractive index). Consequently, this effect confines the radiative flux toward the center so that it increases the temperature in this region, whereas the temperature field around the boundary is less important than the one encountered in the homogeneous medium.

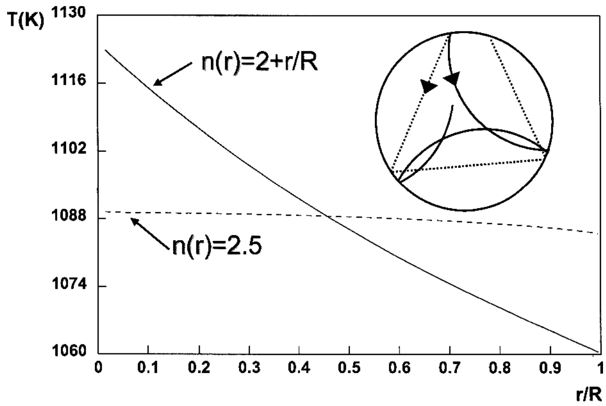

Fig. 4 Illustration of the global curvature effect on the temperature field when the rays bend toward the cylinder center (T_R = 1000 K; R = 10^{-1} m; dissipated power $\mathcal{P}$ = 5.10^5 Wm^{-3} and κ = 10^{-1} m^{-1}).

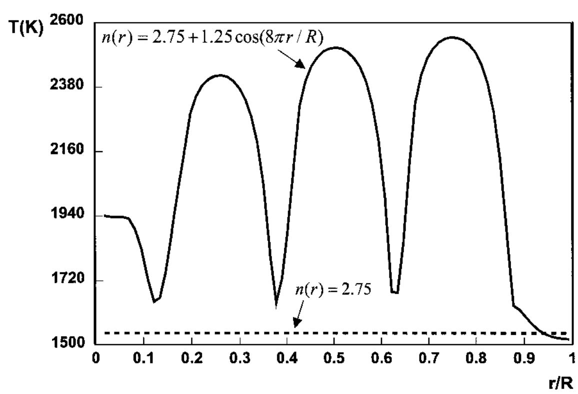

Fig. 5 Illustration of the local curvature effect on the temperature field. The local confinement of many rays causes overheating (T_R = 1500 K; R = 10^{-2} m; $\mathcal{P}$ = 6.10^6 Wm^{-3} and κ = 1 m^{-1}).

Case B: Confinement Effects

We consider a sinusoidal refractive index variation [$n = 2.75 + 1.25\cos(8\pi r/R)$]. The analysis of h_α^β roots shows the existence of three confinement areas within the cylinder. Therefore, for many origin points r_0 and many polar directions γ_0 some confined trajectories exist. The temperature field represented in Fig. 5 proves that this phenomenon is responsible for local overheating. Around the boundary the refractive index is an increasing function so that the ray (and the flux lines) bends to the center of cylinder. This effect is similar to the one encountered in case A. It explains that the temperature dies away from $r/R \sim 0.9$ to 1. In particular the temperature is less important than the one of homogeneous medium (in dotted line). On the other hand, close to the center one notes the temperature is greater than the homogeneous case. This effect cannot be attributed to a confinement phenomenon because for any invariant α the function $g_\alpha(r)$ is null at the origin. Therefore this is at least a consequence of the rays curvature. In this region the rays bend toward the cylinder of radius $0.1R$ because n is locally a decreasing function. Otherwise the adjacent confinement area ($r/R \sim 0.1$ to 0.4) heats the core of cylinder. It plays the role of a thermal barrier. The combination of both phenomena is probably responsible of the substantial overheating around the center.

Conclusion

We have derived the integral forms of the radiative transfer equation for any refractive semitransparent cylindrical medium with specularly reflecting boundaries and anisotropic scattering. These forms contain all boundary conditions of the problem and can easily be solved numerically with existing numerical techniques. This paper offers a theoretical framework for future investigation of the coupling between the refractive index and the state variables (temperature or species concentration fields). This problem is of prime importance for studying the inner structure of numerous semitransparent media (flames, plasmas, etc.) and to improve their nonintrusive metrology.

From a fundamental point of view, two thermal effects have been exhibited within the refractive medium. The first one is global and is caused by the curvature of optical paths, whereas the second is a local effect attributable to the confinement of light within many regions. Both phenomena are direct consequences of the nonlinearity of the optical problem and might occur simultaneously within a medium.

From a technological point of view, the radiative flux control is a potential application for these media, which could result in new devices. However, this technology, which is optically close to fiber optics technology, needs numerous feasibility studies intimately related to the synthesis of refractive materials, before being implemented.

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